HW10.

Problems: 3.4.2, 3.4.3, 3.4.9, 4.3.1.

Problem 3.4.2.

: Determine the change-of-basis matrix from $(e_1 \dots e_n)$ to $(e_n \dots e_1)$.

Solution. The $n \times n$ anti-diagonal matrix $\begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}$.

Solved.

Problem 3.4.3.

: Determine the change-of-basis matrix from (e_1, e_2) to (e_1+e_2, e_1-e_2).

Solution. The matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Solved.

HW10 cont.

Problem 3.4.9.

: Prove that $GL_n(\mathbb{F})$ and the set of bases of $\mathbb{F}_n$ are isomorphic as sets.

Solution. We're going to ~~show~~ identify every basis with its ~~the~~ change-of-basis matrix from the standard basis. Every change-of-basis matrix is clearly invertible, and we'll show that if $A \in GL_n(\mathbb{F})$ then $(Ae_1, Ae_2, \dots, Ae_n)$ is a basis.

Theorem. $(Ae_1, Ae_2, Ae_3, \dots, Ae_n)$ is linearly independent.

Proof. Assume that it is linearly dependent. Then we have

$$c_1 Ae_1 + c_2 Ae_2 + \dots + c_n Ae_n = A(c_1 e_1 + \dots + c_n e_n) = 0,$$

where $\{c_n \mid n \in \mathbb{N}\} \neq \{0\}$, and therefore the Kernel of A is non-trivial, contradicting invertibility.

Qed.

Theorem. $(Ae_1, \dots, Ae_n)$ is spanning for $\mathbb{F}^n$ and thus a basis.

Proof. It is linearly independent and has n elements.

Qed.

By these propositions, every ~~basis~~ invertible matrix uniquely corresponds to a basis of $\mathbb{F}^n$.

Solved.

HW10 cont.

<u>Problem 4.3.1.</u>

: Let V be the vector space of 2×2 symmetric matrices, and let $A=\begin{bmatrix}2&1\\ &1\end{bmatrix}$. Determine the matrix for the operator $X\mapsto AXA^T$ w.r.t. a suitable basis of V.

Solution. Take the following basis of V: $\left(\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)$.
By computation, one can verify the following:

$$v_1=\begin{bmatrix}1&0\\0&0\end{bmatrix}\mapsto\begin{bmatrix}4&0\\0&0\end{bmatrix}=4v_1$$

$$v_2=\begin{bmatrix}0&1\\1&0\end{bmatrix}\mapsto\begin{bmatrix}4&2\\2&0\end{bmatrix}=4v_1+2v_2$$

$$v_3=\begin{bmatrix}0&0\\0&1\end{bmatrix}\mapsto\begin{bmatrix}1&1\\1&1\end{bmatrix}=v_1+v_2+v_3$$

which yields the transformation matrix below.

$$\begin{bmatrix}4&4&1\\0&2&1\\0&0&1\end{bmatrix}=\left[X\mapsto AXA^T\right].$$

Solved.