HW 4.

Problem X2.

: Let $\mathfrak{K}_4$ denote the Klein 4-group. Prove $\operatorname{Aut} \mathfrak{K}_4 \cong S_3$.

Solution. Let's write the mul. table for $\mathfrak{K}_4$:

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

Clearly $\mathfrak{K}_4$ is Abelian and any automorphism has to send 1 to 1, but is free to permute the other 3 elements.

Therefore, $\operatorname{Aut}(\mathfrak{K}_4) \cong \operatorname{Sym}\{1,2,3\}$.

Solved.

HW 4 cont.

Problem X3.

: Define $f : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ by $f(A) = (A^T)^{-1}$. Show that f is an automorphism, but not an inner automorphism.

Solution. It is clear that f is a homomorphism, as

$$f(AB) = AB^{-T} = (B^T A^T)^{-1} = A^{-T}B^{-T} = f(A)f(B).$$

is satisfied. It is self-inverse, so an automorphism.

Theorem. f is not an inner automorphism.

Proof. Assume that it is, i.e. there is a $B \in GL_n(\mathbb{R})$ s.t. for any matrix A we have $B^{-1}AB = A^{-T}$.

Multiplying on the left by B, on the right by A^T, we get the unlikely-looking eqn.

$$ABA^T = B,$$

for any matrix $A \in GL_n(\mathbb{R})$.

Now take

$$A = \begin{bmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{bmatrix},$$ note that $A = A^T$.

Calculating ABA, assuming $ABA = B$, we get that the first and the last column of B are the same, therefore $B \in GL_n(\mathbb{R})$ but $\det B = 0$.

Qea. QEA abbreviates "Quod Est Absurdum". It is used to conclude proofs by contradiction.

Solved.

HW 4 cont.

<u>Problem 1.4.5.</u>

: Prove that the transpose of a permutation matrix is its inverse.

Solution. Let's calculate the (i,j) coefficient of AA^T:

$$(AA^T)_{ij} = \sum_k A_{ik} A^T_{kj} = \sum_k A_{ik} A_{jk}.$$

Now, $A_{ik} A_{jk}$ is 1 iff. the permutation sends i to k and j to k. This occurs only if $i=j$ as permutations are invertible.

Therefore

$AA^T_{ij} = 1$ iff. $i=j$, otherwise $AA^T_{ij} = 0$,

so $AA^T = 1$, which was to be shown.

Solved.